

\mathbb{Q} -divisors

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Let X be a projective scheme.

A \mathbb{Q} -divisor on X is an element of

$$\text{Div}_{\mathbb{Q}}(X) =_{\text{def}} \text{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

i.e. $D \in \text{Div}_{\mathbb{Q}}(X)$ can be written

$$D = \sum c_i A_i, \quad A_i \text{ prime}, \quad c_i \in \mathbb{Q}$$

D is effective if $c_i \geq 0$.

Properties

i.) Intersection works as expected (extension of scalars)

ii.) $D_1, D_2 \in \text{Div}_{\mathbb{Q}}(X)$ are numerically equivalent

$$D_1 \equiv_{\text{num}} D_2$$

if $D_1 \cdot C = D_2 \cdot C$ for every curve $C \in X$.

iii.) $D_1 \equiv_{\text{lin}, \mathbb{Q}} D_2$ if $\exists r \in \mathbb{Z}$ s.t. $rD_1, rD_2 \in \text{Div}(X)$ and

$$rD_1 \equiv_{\text{lin}} rD_2$$

Caution: This is not the standard definition of linear equivalence.

(Take C hyperelliptic, $P, Q \in C$ s.t. $2P \equiv 2Q$, but $P \not\equiv Q$

using standard def of linear equivalence.) We mostly will be concerned w/ the numerical class of a \mathbb{Q} -divisor.

iv.) $f: X \rightarrow Y$ a morphism of projective varieties. This induces a morphism $f^*: N'(X)_{\mathbb{Q}} \rightarrow N'(Y)_{\mathbb{Q}}$ by extension of scalars.

Amplitude

$D \in \text{Div}_{\mathbb{Q}}(X)$ is ample if any of the following equivalent statements are satisfied:

i.) $D = \sum c_i A_i$, where $c_i \in \mathbb{Q}_{>0}$ and A_i is an ample Cartier divisor.

ii.) $\exists r \in \mathbb{Z}_{>0}$ st. $r \cdot D \in \text{Div}_{\mathbb{Z}}(X)$ is ample.

iii.) D satisfies Nakai's criterion. i.e.

$$D^{\dim V} \cdot V > 0$$

for all irreducible $V \subseteq X$ of positive dimension.

Note: From this definition, we can see that amplitude is an open condition, i.e.:

Prop: $H, E \in \text{Div}_{\mathbb{Q}}(X)$ such that H is ample. Then $H + \varepsilon E$ is ample for sufficiently small $\varepsilon \in \mathbb{Q}_{>0}$.

Pf: We can assume H, E are integral by replacing w/ aH, aE

where $a \in \mathbb{Z}_{>0}$.

Then choose m_1, m_2 so that mH is very ample for $m \geq m_1$, and $mH + E$ is globally generated for $m \geq m_2$.

Then if $m \geq m_1 + m_2$, $mH + E$ is very ample, so $H + \frac{1}{m}E$ is ample. \square

(Note: this is also true for finitely many \mathbb{Q} divisors E_1, \dots, E_r)

\mathbb{R} -divisors

$\text{Div}_{\mathbb{R}}(X) = \text{Div}(X) \otimes \mathbb{R}$. All definitions analogous.

$D \in \text{Div}_{\mathbb{R}}(X)$ is ample if $D = \sum c_i A_i$, $c_i > 0$ real and A_i an ample Cartier divisor.

Nakai criterion: If $D \in \text{Div}_{\mathbb{R}}(X)$ is ample, then

$$(D^{\dim(V)} \cdot V) > 0$$

for all irreducible $V \subseteq X$ of positive dimension.

Note: It's not obvious that the converse holds! But it does. Pf requires more machinery.

Thus, amplitude depends only upon numerical equivalence class.

Proposition: Amplitude of \mathbb{R} -divisors is an open condition (proof is similar to \mathbb{Q} case)

Cor: $N^1(X)_{\mathbb{R}}$ is spanned by ample divisors.

Nef \mathbb{Q} - and \mathbb{R} -divisors

$D \in \text{Div}_{\mathbb{R}}(X)$ is nef if $D \cdot C \geq 0$ for all irreducible curves $C \subset X$.

Note: L globally generated \Rightarrow nef.

Some properties of nef bundles/divisors:

1.) $f: Y \rightarrow X$ proper. If L is nef, then $f^*(L)$ is nef.

2.) If in addition f is surjective, then the converse holds. i.e. $f^*(L)$ nef $\Rightarrow L$ nef.

3.) L globally generated $\Rightarrow L$ nef.

4.) If $D \subseteq X$ effective, then $N_{D/X}$ nef $\iff D$ is nef.

Note: 4.) implies that if X is a surface and D effective, then D nef $\iff D^2 \geq 0$.

Kleiman's Theorem: If D is a nef \mathbb{R} -divisor on X ,

then $D^k \cdot V \geq 0$ for every irreducible $V \subseteq X$ of dim k .

Idea of proof: Induction on $n = \dim X$. Assume true for $\dim X < n$

Then, since restriction to subschemes preserves nefness,
 $D^k \cdot V \geq 0 \quad \forall k < n$, so just need to show $D^n \geq 0$.

Cor: Let H be an ample \mathbb{R} -divisor, then $D + \varepsilon H$ is ample for
 all $\varepsilon > 0 \iff D$ is nef.

Pf: If $D + \varepsilon H$ is ample, then $D \cdot C + \varepsilon H \cdot C > 0 \quad \forall$
 irreducible curves $C \subset X$. Taking $\varepsilon \rightarrow 0$, we get $D \cdot C \geq 0$.

Conversely, assume D is nef. Fix $V \subset X$ irreducible, $\dim k > 0$.

$$\text{Then } (D + \varepsilon H)^k \cdot V = \sum_{s=0}^k \binom{k}{s} \varepsilon^s (H^s \cdot D^{k-s}) \cdot V. \quad (*)$$

Since H is a positive \mathbb{R} -linear combination of integral ample
 divisors, $H^s \cdot V$ can be represented by a positive \mathbb{R} -linear
 combination of $(k-s)$ -divisors.

So by the theorem, the summands of $(*)$ are ≥ 0 for
 $s < k$, and $H^k \cdot V > 0$, so $(*) > 0$, as desired \square

Cor: X projective variety, D and H \mathbb{R} -divisors, H ample.

Then D is ample if and only if $\exists \varepsilon > 0$ s.t.

$$(*) \quad \frac{(D \cdot C)}{(H \cdot C)} \geq \varepsilon \quad \text{for every irreducible curve } C \subseteq X.$$

Pf: $(*) \iff (D - \varepsilon H)$ is nef. Thus, if $(*)$ holds, $D - \varepsilon H + \varepsilon H = D$ is

ample by previous corollary (possibly choosing smaller ε).

If D is ample, then $D - \varepsilon H$ is ample for suff. small $\varepsilon > 0$. \square